

Goldman flows on the Jacobian

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Abstract

We show that the Goldman flows preserve the holomorphic structure on the moduli space of homomorphisms of the fundamental group of a Riemann surface into $U(1)$, in other words the Jacobian.

1 Introduction

This note concerns the moduli space $\mathcal{M}(G)$ of conjugacy classes of homomorphisms of the fundamental group of a compact orientable 2-manifold Σ into a Lie group G .

This object has recently attracted a great deal of interest in symplectic and algebraic geometry and mathematical physics. In mathematical physics it appears as the space of gauge equivalence classes of flat connections on a 2-manifold. In algebraic geometry it appears as the moduli space of holomorphic bundles on a Riemann surface.

The smooth locus of the space $\mathcal{M}(G)$ has a symplectic structure; see, for instance, [1] or [7]. If the 2-manifold is equipped with a complex structure, the space $\mathcal{M}(G)$ inherits a complex structure compatible with the symplectic structure. When $G = U(1)$, the moduli space of a Riemann surface coincides with its Jacobian, which is a complex torus whose complex dimension is the genus of the surface.

In [2], W. Goldman studied the Hamiltonian flows of certain natural functions on the moduli space. These functions are constructed from functions that send a flat connection to its holonomy around a specific simple closed curve C in the 2-manifold.

In this paper we show that when the gauge group is $U(1)$ the Goldman flows preserve the complex structure on $\mathcal{M}(U(1))$. When the gauge group is $SU(2)$, the Goldman flows are ill defined on a set of real codimension 3, (see [6]), which is inconsistent with these flows preserving the complex structure.

2 The $U(1)$ Goldman flow

Let $G = U(1)$. The Goldman flow on $\mathcal{M}(G)$ is a periodic \mathbb{R} -action $\{\Xi_s\}_{s \in \mathbb{R}}$ associated to a simple closed curve C . Since the Lie group is abelian, the moduli space is $\text{Hom}(\pi, G)$.

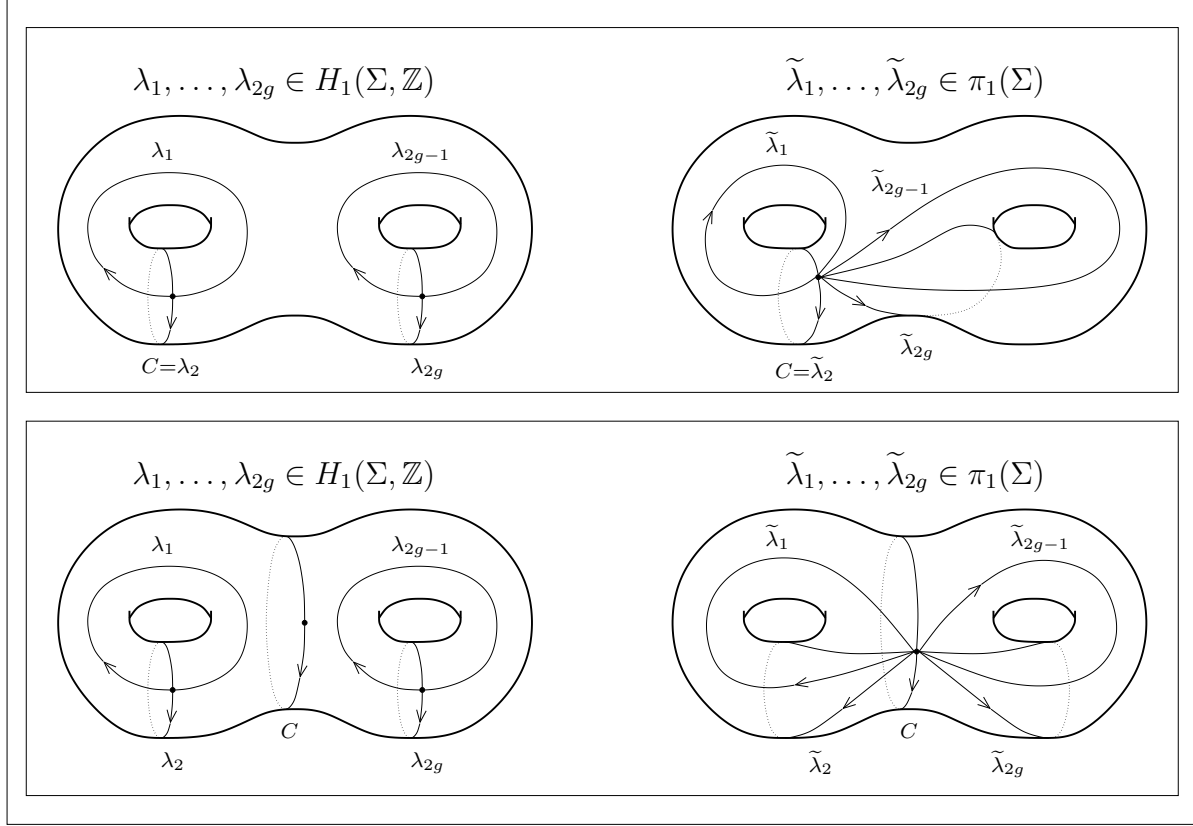


Figure 1: The curve C is either nonseparating (top), or separating (bottom).

Choose a symplectic basis of $H_1(\Sigma, \mathbb{Z})$, in other words a collection of cycles $\{\lambda_1, \dots, \lambda_{2g}\}$ in which all the intersections are empty except for λ_{2j-1} and λ_{2j} , which intersect transversely with positive intersection index. If the curve C is nonseparating then we let $\lambda_2 = C$ in this symplectic basis; if the curve C is separating then we assume that the cycles $\lambda_1, \dots, \lambda_{2g}$ do not intersect C . Choose a basepoint on C for the fundamental group of Σ , and lift the cycles in the symplectic basis to loops $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{2g} \in \pi_1(\Sigma)$ that only intersect C at their endpoints. See Figure 1.

For $G = U(1)$, identify $\Phi \in \text{Hom}(\pi_1(\Sigma), G)$ with $(\Phi_1, \dots, \Phi_{2g}) \in G^{2g}$ by letting $\Phi_j = \Phi(\tilde{\lambda}_j)$. If the simple closed curve C is nonseparating then the associated Goldman flow on $\text{Hom}(\pi, G) = G^{2g}$ is

$$\Xi_s(\Phi) = (e^{2\pi i s} \Phi_1, \Phi_2, \dots, \Phi_{2g}),$$

for $s \in \mathbb{R}/\mathbb{Z}$. If C is separating then the Goldman flow is trivial, $\Xi_s(\Phi) = \Phi$.

The Goldman flows can be described using gauge theory as follows, (cf. [3]). If A is a flat connection on the trivial G -bundle over Σ , and A_j is the holonomy of A along the loop $\tilde{\lambda}_j$, then the map that sends A to $(A_1, \dots, A_{2g}) \in G^{2g} = \text{Hom}(\pi_1(\Sigma), G)$ identifies the space of gauge equivalence classes of flat connections with the moduli space; see [5]. Let $\hat{\Sigma}$ be the complement of C in Σ , and let $U_- \cup U_+$ be the intersection of $\hat{\Sigma}$ with a tubular

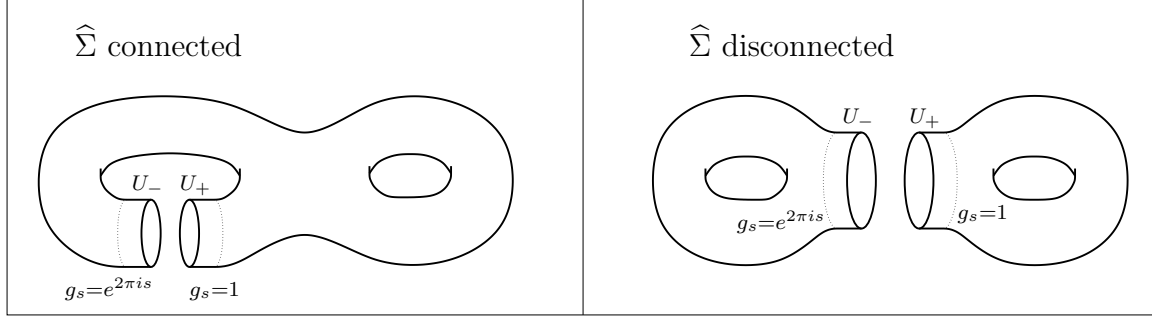


Figure 2: The gauge transformation g_s .

neighbourhood of C in Σ . The open sets U_- and U_+ can be thought of as neighbourhoods of the two “boundary components” of $\widehat{\Sigma}$, as shown in Figure 2. For a flat connection A on Σ , define

$$\Xi_s(A) = A^{g_s},$$

where g_s is a gauge transformation on $\widehat{\Sigma}$ with $g_s = 1$ on U_+ and $g_s = e^{2\pi is}$ on U_- . Here, since g_s is constant on both U_+ and U_- and since the gauge group is abelian, the flat connection A^{g_s} on $\widehat{\Sigma}$ extends (by A) to a flat connection $\Xi_s(A)$ on Σ . If the curve C separates Σ into two components then $\Xi_s(A) = A$ because g_s may be chosen to be a locally constant gauge transformation on $\widehat{\Sigma}$, which acts trivially since G is abelian. If the curve C is nonseparating, however, then the gauge transformations g_s on $\widehat{\Sigma}$ act nontrivially and do not come from gauge transformations on Σ for $s \notin \mathbb{Z}$, so in general the $\Xi_s(A)$ are distinct elements of the moduli space $\mathcal{M}(G)$, (although they are the same when viewed as elements of the moduli space of $\widehat{\Sigma}$).

3 The Jacobian

When $G = U(1)$, the moduli space $\mathcal{M}(U(1))$ is the Jacobian $\text{Jac}(\Sigma) \cong U(1)^{2g}$. The Jacobian inherits a complex structure from the Riemann surface Σ , and identifies with \mathbb{C}^g/Λ as a complex manifold for a lattice Λ described below. See, for example, [4].

The Jacobian is defined as

$$\text{Jac}(\Sigma) = \frac{H^0(\Sigma, K)^*}{H_1(\Sigma, \mathbb{Z})},$$

where $H_1(\Sigma, \mathbb{Z})$ maps to $H^0(\Sigma, K)^*$ by integration: a class $\lambda \in H_1(\Sigma, \mathbb{Z})$ sends $\omega \in H^0(\Sigma, K)$ to $\int_\lambda \omega \in \mathbb{C}$. Explicitly, choose a basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(\Sigma, K)$, and use the dual basis to identify $H^0(\Sigma, K)^*$ with \mathbb{C}^g . Let F be the resulting map from $H_1(\Sigma, \mathbb{Z})$ to \mathbb{C}^g ,

$$F(\lambda) = \begin{pmatrix} \int_\lambda \omega_1 \\ \vdots \\ \int_\lambda \omega_g \end{pmatrix},$$

and equate $H_1(\Sigma, \mathbb{Z}) \subset H^0(\Sigma, K)^*$ with the lattice

$$\Lambda = \{F(\lambda) : \lambda \in H_1(\Sigma, \mathbb{Z})\} \subset \mathbb{C}^g.$$

A choice of basis $\{\lambda_1, \dots, \lambda_{2g}\}$ of $H_1(\Sigma, \mathbb{Z})$ identifies \mathbb{C}^g/Λ with $U(1)^{2g}$ as follows. Viewed as a *real* vector space, \mathbb{C}^g is spanned by $\{F(\lambda_1), \dots, F(\lambda_{2g})\}$,

$$\mathbb{C}^g = \left\{ \sum_{j=1}^{2g} v_j F(\lambda_j) : v_j \in \mathbb{R} \right\}.$$

Identify \mathbb{C}^g/Λ with $U(1)^{2g}$ by the group isomorphism that maps

$$\left[\sum_{j=1}^{2g} v_j F(\lambda_j) \right] \in \mathbb{C}^g/\Lambda$$

to

$$(\exp 2\pi i v_1, \dots, \exp 2\pi i v_{2g}).$$

We define $z_j = \exp 2\pi i v_j$.

4 Goldman flows on the Jacobian

The Goldman flow on the Jacobian is defined as follows. If C is a nonseparating simple closed curve then choose C as the generator λ_2 in a symplectic basis $\{\lambda_1, \dots, \lambda_{2g}\}$ of $H_1(\Sigma, \mathbb{Z})$. So the Goldman flow associated to C is

$$(z_1, z_2, \dots, z_{2g}) \mapsto (e^{2\pi i s} z_1, z_2, \dots, z_{2g})$$

for $s \in \mathbb{R}/\mathbb{Z}$. This corresponds to translation by $sF(\lambda_1)$ in \mathbb{C}^g , which is clearly a holomorphic self map of \mathbb{C}^g/Λ .

If C is a separating simple closed curve then the Goldman flow on the Jacobian is trivial (because the gauge group is abelian).

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